Math 3236 Statistical Theory

4/11/23

$$\frac{b_1}{N\bar{T}} \le \lambda \le \frac{b_2}{N\bar{T}}$$

$$\left(1 - \frac{c}{\sqrt{N}}\right)\frac{1}{\bar{T}} \le \lambda \le \frac{1}{\bar{T}}\left(1 + \frac{c}{\sqrt{N}}\right)$$

$b_1, b_2 \to 1$ as $N \to \infty$

$$b_1 = 1 + \frac{c + \delta_{1,N}}{\sqrt{N}}$$

with

$\delta_{1,N} \xrightarrow[N \to \infty]{} 0$

$$b_2 = 1 - \frac{c + \delta_{2,N}}{\sqrt{N}}$$

$\delta_{2,N} \xrightarrow[N \to \infty]{} 0$

$X_i \quad i = 1 \ldots N$

$f(x_i; \theta)$ p.d.f. of X_i

$\theta \in \Omega$ parameter.

$\Omega = \Omega_0 \cup \Omega_1$

$\Omega_0 \cap \Omega_1 = \phi$

$H_0 = \theta \in \Omega_0$

$H_1 \, (H_a) : \theta \in \Omega_1$

Ex. 1

$X_1 \sim N(\mu, \sigma^2) \quad \sigma$ Known

$H_0 : \mu = \mu_0$ (simple hyp.)

Two sided

$H_a : \mu \neq \mu_0$ (composite hyp)

$H_0 : \mu \leq \mu_0$

$H_a : \mu > \mu_0$

one sided Test

Ex 2 Uniform dist.

X_i uniform in $[0, \theta]$

$H_0 = 3 \le \theta \le 4$

$H_a = \theta < 3$ or $\theta > 4$

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§ Test procedure.

S is the sample space for the r.s. $\underline{X}$,

$S_0 \cup S_1 = S$ $\qquad$ $S_0 \cap S_1 = \emptyset$

If $\underline{x}$ is a realization $\underline{X}$

Then

$\underline{x} \in S_0$ $\qquad$ do not reject H_0

$\underline{x} \in S_1$ $\qquad$ reject H_0

S_1 "critical region"

δ is normally based on a statistics T

δ: reject if $T \geq c$

for some c.

$R = \{c, +\infty\}$ rejection region

T Test statistics.

$\Omega = \Omega_0 \cup \Omega_1$

$S = S_0 \cup S_1$

(T Test statistics

$S_1 = T^{-1}(R)$)

Eventually, you perform your sample and learn whether $\underline{x} \in S_0$ or not.

You'll never know with certainity if $\theta \in \Omega_0$ or not.

Power function of a test

$\pi(\delta \mid \theta_0)$ probability of rejecting H_0 when $\theta = \theta_0$

$$\mathbb{P}(\underline{X} \in S_1 \mid \theta_0)$$

Ideally

$$\pi(\delta \mid \theta) = 0 \qquad \theta \in S_0$$

$$\pi(\delta \mid \theta) = 1 \qquad \theta \in S_1$$

$\pi(\delta \mid \theta)$ as small as possible when $\theta \in S_0$

$\pi(\delta, \theta)$ as large as possible when $\theta \in S_1$

X_i are normal μ, σ^2

σ^2 Known

$H_0 : \mu = \mu_0$

$H_a : \mu \neq \mu_0$

$$\bar{X} = \frac{1}{N} \sum_{i=1}^{N} X_i$$

if H_0 is True, I expect

$$\bar{x} \simeq \mu_0$$

$$T = |\bar{X} - \mu_0|$$

δ : $T \geq c$ reject H_0

$$\mathbb{P}(T \in R \mid \mu) = \mathbb{P}(\bar{X} - \mu_0 \leq -c \mid \mu)$$
$$+ \mathbb{P}(\bar{X} - \mu_0 \geq c \mid \mu)$$

$$= \mathbb{P}(\bar{X} \geq \mu_0 + c \mid \mu) +$$

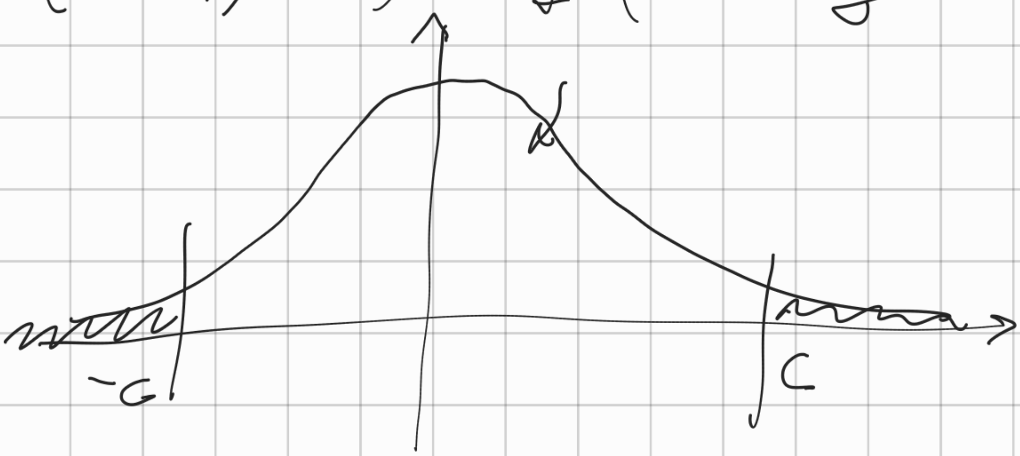
$$\mathbb{P}(\bar{X} \leq \mu_0 - c \mid \mu)$$

$$= 1 - \Phi\left(\sqrt{N}\,\frac{\mu_0 + c - \mu}{\sigma}\right) +$$

$$\Phi\left(\sqrt{N}\,\frac{\mu_0 - c - \mu}{\sigma}\right) =$$

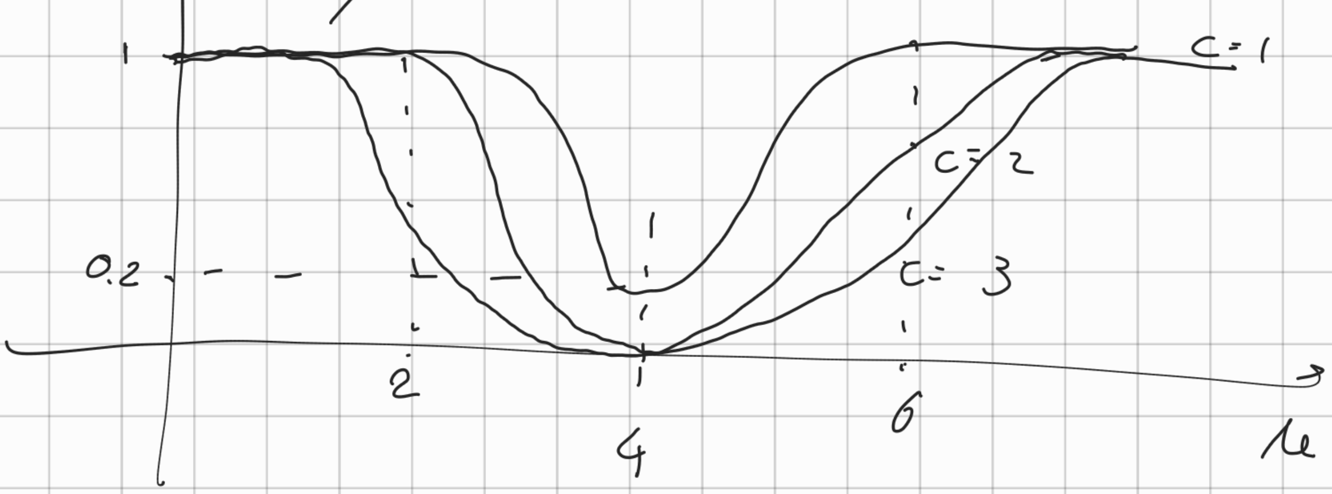
$$\pi(\delta \mid \mu)$$

$$\pi(\delta \mid \mu_0) = 2\,\Phi\left(-\frac{c\sqrt{N}}{\sigma}\right)$$



If I want $\pi(\delta \mid \mu_0)$ be small I need c large

$\mu_0 = 4 \quad n = 15 \quad \sigma^2 = 9$



X_i uniform in $[0, \theta]$

$H_0 : 3 \leq \theta \leq 4$

$H_a : \theta < 3$ or $\theta > 4$

$Y = \max\{X_1, \ldots X_n\}$

if $Y > 4$

δ : do not reject H_0 if

$2.9 \leq Y \leq 4$

if $\theta \leq 2.9 \implies$

$Y < 2.9$ prob 1

$\pi(d, \theta) = 1$ if $\theta \leq 2.9$

if $2.9 \leq \theta \leq 4$

$$\mathbb{P}(Y < 2.9 \mid \theta) = \left(2.9/\theta\right)^N$$

$$\mathbb{P}(Y > 4 \mid \theta) = 0$$

if $\theta \geq 4$

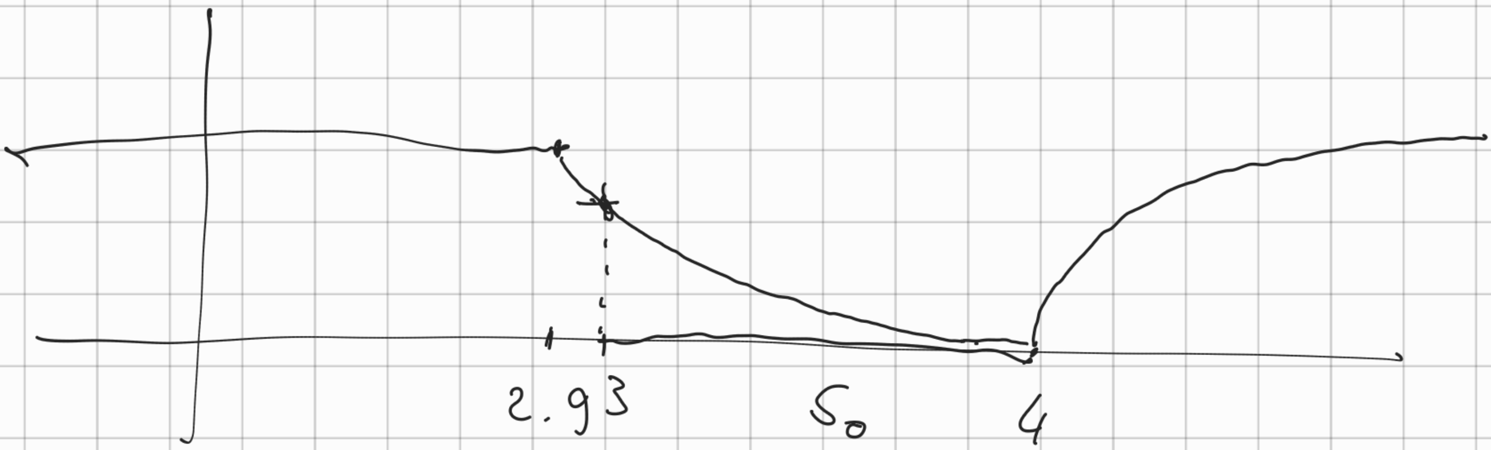
$$\mathbb{P}(Y \leq 2.9 \mid \theta) = \left(2.9/\theta\right)^N$$

$$\mathbb{P}(Y \geq 4 \mid \theta) = 1 - \left(4/\theta\right)^N$$

$$\pi(d \mid \theta) = 1 \qquad \theta \leq 2.9$$

$$= \left(2.9/\theta\right)^N \qquad 2.9 \leq \theta \leq 4$$

$$= 1 - \left(4/\theta\right)^N + \left(2.9/\theta\right)^N \qquad \theta \geq 4$$



$\alpha(\delta) := \sup_{\theta \in S_0} \pi(\delta \mid \theta)$ size of the Test

Ex 1 Normal r.v.

$H_0 : \mu = \mu_0$

$$\alpha(\delta) = \pi(\delta \mid \mu_0)$$

Ex 2. $\alpha(\delta) = \pi(\delta \mid 3) = \left(2.9/3\right)^N$

We say that a Test δ has significance level α if

$\alpha(d) \leq \alpha$.